

A FEW REMARKS ON QUADRATIC HARNESES

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ABSTRACT. We analyze and partially solve system of recurrences that can be derived from the properties of martingale orthogonal polynomials that characterize quadratic harnesses (QH). On the way we simplify proofs that were contained in original derivation of the system done by W. Bryc, W. Matysiak and Wołowski correcting mistakes and misprints. We give also specify conditions for the existence of moments of one dimensional distribution for large classes of quadratic harnesses that are also Markov processes complementing results recently obtained by W. Bryc.

1. INTRODUCTION

In the series of papers [2], [5], [6], [3], [7], [8], [10], [9], Bryc and Wołowski supported from time to time by Matysiak define and analyze a wide class of stochastic processes that they call Quadratic Harnesses (briefly QH).

In this paper we concentrate on a subclass of QH that are also Markov processes (briefly MQH) and possess a sequence of orthogonal polynomials that was called by Bryc, Matysiak and Wołowski the sequence of martingale polynomials (briefly sequence of OMP). We concentrate on the properties of these polynomials and consequently deduce important properties of the related MQH.

In doing so we correct mistakes and misprints that were made in formulae (2.22)-(2.25) of [6]. In particular we present detailed recurrences that are to be satisfied by 6 families of number sequences that define, together with parameters of QH, sequence of OMP. We also derive some conditions expressed in terms of parameters for the existence of all moments and the uniqueness of one-dimensional distributions defined by these moments complementing conditions recently obtained by W. Bryc in [11]. Finally we also consider some special cases of values of the parameters and present their consequences.

We would like to underline that we are not analyzing possibly resulting from our considerations quadratic harnesses. It would be too great task for the 15 pages paper. Besides Bryc, Matysiak and Wołowski do this perfectly well and we are not going to compete with them. Rather to we are complementing their results.

The paper is organized as follows. In the next section we recall the definition of QH following works of Bryc and others. In particular we derive system of 5 iterative equations that must be satisfied if sequence of OMP is to exist for a given QH.

In the next Section 2 we study these equations partially solving them or at least providing conditions for the existence of the identifiable by moments one dimensional measure. On the way we use results of special auxiliary Section 5

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that we have placed at the end of the paper. The third Section contains our main results concerning integrability and existence of sequence of OMP for a given QH characterized by 5 parameters. Next short Section 4 contains some remarks concerning the results and presents some open problem that appeared while writing the paper. Final, fifth Section 6 contains uninteresting or lengthy proofs.

Throughout the paper we use traditional notation used in q -series theory. In particular we denote:

$$[0]_q = 0; [n]_q = 1 + q + \dots + q^{n-1}, [n]_q! = \prod_{j=1}^n [j]_q,$$

with $[0]_q! = 1$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q! [k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

It is also useful to use the so called q -Pochhammer symbol for $n \geq 1$:

$$\begin{aligned} (a; q)_n &= \prod_{j=0}^{n-1} (1 - aq^j), \\ (a_1, a_2, \dots, a_k; q)_n &= \prod_{j=1}^k (a_j; q)_n. \end{aligned}$$

with $(a; q)_0 = 1$. Often $(a; q)_n$ as well as $(a_1, a_2, \dots, a_k | q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \dots, a_k)_n$, if it will not cause misunderstanding.

2. QUADRATIC HARNESES

Let us recall, following [6] that QH is a stochastic process $\{X_t\}_{t \geq 0}$ defined for $t \geq 0$ on a certain probability space (Ω, \mathcal{F}, P) satisfying the following definition:

Definition 1. *A stochastic process $\{X_t\}_{t \geq 0}$ will be called quadratic harness if the following 4 conditions are satisfied:*

1. $X_0 = 0, \forall t \geq 0, EX_t = 0$,
2. $\forall s, t \geq 0, EX_s X_t = \min(s, t)$,
3. $\forall 0 \leq s < t < u : E(X_t | \mathcal{F}_{s,u}) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u, a.s.$
4. $\forall 0 \leq s < t < u : E(X_t^2 | \mathcal{F}_{s,u}) = Q_{s,t,u}(X_s, X_u)$,

where $Q_{s,t,u}(x, y)$ is a certain quadratic form determined by 6 coefficients and $\mathcal{F}_{s,u} = \sigma(X_t : t \in (0, s] \cup [u, \infty))$.

Bryc, Matysiak, Wołowski showed in [6] that there exist 5 parameters which they denoted by $\tau, \sigma, \theta, \eta, q$ such that the quadratic form Q is completely determined i.e. respective coefficients are defined by the known functions of s, t, u and $\tau, \sigma, \theta, \eta, q$. Bryc, Matysiak, Wołowski deduced that $\sigma, \tau \geq 0, q \leq 1 + 2\sqrt{\sigma\tau}$ and $\eta, \theta \in \mathbb{R}$. More precisely they showed that

$$Q_{s,t,u}(x, y) = A(s, t, u)x^2 + B(s, t, u)xy + C(s, t, u)y^2 + D(s, t, u)x + E(s, t, u)y + F(s, t, u),$$

where

$$(2.1) \quad A(s, t, u) = \frac{(u-t)(u(1+\sigma t) + \tau - qt)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(2.2) \quad B(s, t, u) = \frac{(u-t)(t-s)(1+q)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(2.3) \quad A(s, t, u) = \frac{(t-s)(t(1+\sigma s) + \tau - qs)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(2.4) \quad D(s, t, u) = \frac{(u-t)(t-s)(u\eta - \theta)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(2.5) \quad E(s, t, u) = \frac{(u-t)(t-s)(-s\eta + \theta)}{(u-s)(u(1+\sigma s) + \tau - qs)},$$

$$(2.6) \quad F(s, t, u) = \frac{(u-t)(t-s)}{(u(1+\sigma s) + \tau - qs)}.$$

The authors seek quadratic harnesses that are also Markov processes and assuming the existence of all moments they try to find a family of orthogonal polynomials $\{p_n(x; t)\}_{t \geq 0, n \geq -1}$ such that

$$(2.7) \quad \forall n \geq 0, t > s \geq 0 : E(p_n(X_t; t) | \mathcal{F}_{\leq s}) = p_n(X_s; s), \text{ a.s.}$$

Such family of QH that are also Markov will be called MQH and obviously they constitute a subset of all QH.

Family of orthogonal polynomials of MQH will be called orthogonal martingale polynomials (briefly OM family of polynomials of the MQH $\{X_t\}$).

Obviously we have $p_{-1}(x, t) = 0$, $p_0(x; t) = 1$. Moreover the authors show in [6] that $p_1(x; t) = x$. Now recall that following general theory of orthogonal polynomials presented e.g. in [15] that every family of orthogonal polynomials $\{r_n(x)\}$ satisfies the so called 3-term recurrence, i.e. the product $xr_n(x)$ is a linear combination of r_k for $k = n+1, n, n-1$. Moreover coefficients of this linear combination have important meaning. Hence for all $t \geq 0$, $n \geq 0$ we have

$$(2.8) \quad xp_n(x; t) = a_n(t)p_{n+1}(x; t) + b_n(t)p_n(x; t) + c_n(t)p_{n-1}(x; t),$$

where functions a_n , b_n and c_n must depend on t and five parameters $\sigma, \tau, \eta, \theta$ and q only.

Note that if (2.8) and (2.7) are to make sense we must have $a_n(t) > 0$ for all t and $n > -1$. Moreover from the general theory of orthogonal polynomials it follows that if $a_n(t)c_n(t) \geq 0$ for all n then the measure with respect to which polynomials p_n are to be orthogonal is nonnegative i.e. polynomials have probabilistic interpretation. Hence it is reasonable to consider only such QH for which this condition is satisfied for all $n > -1$ and $t \geq 0$.

Bryc, Matysiak, Wesolowski showed also in the same paper that coefficients a_n , b_n , c_n must be linear functions of t . This an easy conclusion of the condition 3. of the Definition 1. For the sake of completeness we will prove this fact.

Proposition 1. *Let $\{X_t\}_{t \geq 0}$ be MQH with parameters $\sigma, \tau, \theta, \eta, q$ such that $\forall t > 0$: supp X_t contains infinite number of points. Let $\{p_n(x; t)\}_{n \geq 0}$ denote its family of OM polynomials. Then*

i) $\forall n > 0$ $p_n(0, 0) = 0$ consequently $\forall n > 0$: $E(p_n(X_t; t)) = 0$ thus polynomials p_n constitute the family of orthogonal polynomials of the marginal distribution i.e. distribution of X_t .

ii) *There must exist six number sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\varepsilon_n\}$, $\{\varphi_n\}$ such that:*

$$a_n(t) = \alpha_n t + \beta_n, \quad b_n(t) = \gamma_n t + \delta_n, \quad c_n(t) = \varepsilon_n t + \varphi_n,$$

with $\alpha_0 = 0$, $\beta_0 = 1$, $\gamma_0 = 0$, $\delta_0 = 0$, $\varepsilon_1 = 1$, $\varphi_1 = 0$.

Proof. Is shifted to Section 6. \square

Hence these coefficients are defined in fact by 6 families of sequences. More precisely we will seek relationships between families of numbers that are implied by the conditions that MQH's must satisfy.

We have the following theorem that corrects mistakes and misprints that can be found in [6]. More precisely in formulae (2.22)-(2.26). In particular formula (2.22) should be divided by σ , in formula (2.23) $\sigma\alpha_{n+1}$ should be replaced by α_{n+1} , in formula (2.24) there is no term $\sigma\alpha_n\varphi_n$ but only $\varphi_n\alpha_{n-1}$ as in (2.13) and (2.14). Further in formula (2.25) the term $q\delta_{n-1}\varepsilon_n$ should be replaced by $q\delta_{n-1}\varphi_n$ and the term $\varphi_n\gamma_{n-1}$ should be replaced by $\varepsilon_n\delta_{n-1}$. Generally formulae (2.9)-(2.14) contain more symmetries than that of [6]. Besides the proof that we are presenting is much simpler both conceptually and mathematically then the result of in [6]. Nevertheless it is relatively long.

Theorem 1. *Assuming that process $\{X_t\}_{t \geq 0}$ is a MQH with parameters $\sigma, \tau, \theta, \eta, q$ and family of polynomials $\{p_n\}$ constitute its family of om polynomials. Then families of numbers $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\varepsilon_n\}$, $\{\varphi_n\}$ satisfy the following system of 5 recurrences:*

$$(2.9) \quad \tau\alpha_n\alpha_{n+1} + q\alpha_n\beta_{n+1} + \sigma\beta_n\beta_{n+1} = \alpha_{n+1}\beta_n,$$

$$(2.10) \quad \tau\varepsilon_{n-1}\varepsilon_n + q\varepsilon_n\varphi_{n-1} + \sigma\varphi_n\varphi_{n-1} = \varepsilon_{n-1}\varphi_n,$$

$$(2.11)$$

$$\theta\alpha_n + \eta\beta_n + \tau\alpha_n(\gamma_n + \gamma_{n+1}) + \sigma\beta_n(\delta_n + \delta_{n+1}) + q(\alpha_n\delta_{n+1} + \beta_n\gamma_n) = \beta_n\gamma_{n+1} + \alpha_n\delta_n$$

$$(2.12)$$

$$\theta\varepsilon_n + \eta\varphi_n + \tau\varepsilon_n(\gamma_n + \gamma_{n-1}) + \sigma\varphi_n(\delta_{n-1} + \delta_n) + q\varphi_n(\gamma_n + \gamma_{n-1}) = \varepsilon_n(\delta_{n-1} + \delta_n)$$

$$(2.13)$$

$$(2.14) \quad \begin{aligned} &1 + \theta\gamma_n + \eta\delta_n + \tau\gamma_n^2 + \sigma\delta_n^2 + \tau(\alpha_{n-1}\varepsilon_n + \alpha_n\varepsilon_{n+1}) + \sigma(\varphi_n\beta_{n-1} + \beta_n\varphi_{n+1}) \\ &+ q(\gamma_n\delta_n + \beta_{n-1}\varepsilon_n + \alpha_n\varphi_{n+1}) = \gamma_n\delta_n + \beta_n\varepsilon_{n+1} + \varphi_n\alpha_{n-1}. \end{aligned}$$

with initial conditions $\alpha_0 = \gamma_0 = \delta_0 = \varphi_1 = 0$ and $\beta_0 = \varepsilon_1 = 1$.

Proof. Proof of this Theorem is shifted to Section 6 since it is long and requires tedious calculations. \square

3. ANALYSIS AND INTEGRABILITY

Notice that if the OMP sequence is to exist coefficients $a_n(t) \neq 0$ for all n and t . That is sequences $\{\alpha_n\}$ and $\{\beta_n\}$ cannot vanish simultaneously. Taking into account equation (2.9) and the fact that $\beta_0 \neq 0$ and assuming that $\beta_n \neq 0$ for $n > 0$ let us divide both sides (2.9) by $\beta_n\beta_{n+1}$ and denote $d_n = \alpha_n/\beta_n$ we get: $d_{n+1} = \tau d_n d_{n+1} + q d_n + \sigma$ or equivalently

$$d_{n+1} = \frac{\sigma + q d_n}{1 - \tau d_n},$$

with $d_0 = 0$. Let us notice that together with $\sigma = 0$ this initial condition leads to $d_n = 0$ for all $n \geq 1$. Thus let us assume that $\sigma \neq 0$ and denote $\lambda_n = d_n/\sigma$. Writing $d_n = \sigma\lambda_n$ we see that sequence $\{\lambda_n\}_{n \geq 1}$ satisfies equation:

$$(3.1) \quad \lambda_{n+1} = \frac{1 + q\lambda_n}{1 - \sigma\tau\lambda_n}.$$

with $\lambda_0 = 0$.

We can perform similar considerations concerning equation (2.10) this time taking into account that $\varepsilon_1 \neq 0$. Consequently we deduce that $\frac{\varphi_n}{\varepsilon_n} = \tau\lambda_{n-1}$.

Further let us consider equations (2.11) and (2.12). Dividing both sides of (2.11) by β_n and keeping in mind that $\frac{\alpha_n}{\beta_n} = \sigma\lambda_n$ we get:

$$\gamma_{n+1}(1 - \sigma\tau\lambda_n) - \sigma(1 + q\lambda_n)\delta_{n+1} = (\tau\sigma\lambda_n + q)\gamma_n + \sigma(1 - \lambda_n)\delta_n + \theta\sigma\lambda_n + \eta.$$

Dividing both sides of (2.12) by ε_n and keeping in mind that $\frac{\varphi_n}{\varepsilon_n} = \tau\lambda_{n-1}$ we get:

$$\delta_{n+1}(1 - \sigma\tau\lambda_n) - \tau(1 + q\lambda_n)\gamma_{n+1} = \tau(1 + q\lambda_n)\gamma_n + (\sigma\tau\lambda_n - 1)\delta_n + \theta + \tau\lambda_n\eta.$$

Let us denote

$$(3.2) \quad A_n = \begin{bmatrix} 1 - \sigma\tau\lambda_n & -\sigma(1 + q\lambda_n) \\ -\tau(1 + q\lambda_n) & 1 - \sigma\tau\lambda_n \end{bmatrix},$$

$$(3.3) \quad B_n = \begin{bmatrix} q + \sigma\tau\lambda_n & \sigma(1 - \lambda_n) \\ \tau(1 + q\lambda_n) & -1 + \sigma\tau\lambda_n \end{bmatrix},$$

$$(3.4) \quad C_n = \begin{bmatrix} \sigma\lambda_n & 1 \\ 1 & \tau\lambda_n \end{bmatrix}.$$

We have the following new vector form of equations (2.11) and (2.12):

$$(3.5) \quad \begin{bmatrix} \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = A_n^{-1}B_n \begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix} + A_n^{-1}C_n \begin{bmatrix} \theta \\ \eta \end{bmatrix},$$

with $\begin{bmatrix} \gamma_0 \\ \delta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $n \geq 0$.

We have also the following simple observation:

Proposition 2.

$$(3.6) \quad \begin{bmatrix} \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = \sum_{k=0}^n \left(\prod_{j=k+1}^n \Xi_j \right) w_k,$$

where we denoted $w_k = A_k^{-1}C_k \begin{bmatrix} \theta \\ \tau \end{bmatrix}$, $\Xi_k = A_k^{-1}B_k$ for $k > 0$. In (3.6) we set

$$\prod_{k=n+1}^n \Xi_k = I \text{ and } \begin{bmatrix} \gamma_1 \\ \delta_1 \end{bmatrix} = w_0.$$

Proof. By induction we have on the left hand side: $\begin{bmatrix} \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = \sum_{k=0}^n w_k (\prod_{j=k+1}^n \Xi_j)$,

while right hand side is $\Xi_n \sum_{k=0}^{n-1} w_k (\prod_{j=k+1}^{n-1} \Xi_j) + w_n = \sum_{k=0}^{n-1} w_k \prod_{j=k+1}^n \Xi_j + w_n$
 $= \sum_{k=0}^n w_k \prod_{j=k+1}^n \Xi_j.$ \square

Finally let us analyze equation given by (2.13) and (2.14). Taking $\alpha_n = \sigma\lambda_n\beta_n$ and $\varphi_{n+1} = \tau\lambda_n\varepsilon_{n+1}$ and denoting $\chi_n = \beta_{n-1}\varepsilon_n$ we get :

$$(3.7) \quad (q + \sigma\tau - \sigma\tau(1 - \lambda_n)^2)\chi_n + 1 + \theta\gamma_n + \tau\gamma_n^2 + \eta\delta_n + \sigma\delta_n^2 - (1 - q)\gamma_n\delta_n \\ = (1 - \sigma\tau(2\lambda_n + q\lambda_n^2))\chi_{n+1}$$

with $\chi_1 = 1$. Let us now redefine orthogonal polynomials p_n . Namely let us consider new polynomials $M_n(y|t, \sigma, \tau, \theta, \eta, q)$ briefly denoted $M_n(y)$ related to polynomials p_n in the following way:

$$(3.8) \quad M_n(y) = \frac{\prod_{j=0}^{n-1} \beta_j(\sigma\lambda_j t + 1)}{t^{n/2}} p_n(y\sqrt{t}).$$

We have the following simple observation:

Proposition 3. *Polynomials $\{M_n\}$ satisfy the following 3-term recurrence:*

$$(3.9) \quad yM_n(y) = M_{n+1}(y) + (\gamma_n t + \delta_n)M_n(y) + \chi_n(1 + \sigma\lambda_{n-1}t + \tau\lambda_{n-1}/t + \sigma\tau\lambda_{n-1}^2)M_{n-1}(y), \\ \text{with } M_{-1}(y) = 0, M_0(y) = 1.$$

Proof. Multiplying both sides of (2.8) by $\prod_{j=0}^{n-1} a_j(t)/t^{(n+1)/2}$ we get

$$\prod_{j=0}^{n-1} a_j(t) \frac{x}{\sqrt{t}} \frac{p_n(x;t)}{t^{n/2}} = \prod_{j=0}^n a_j(t) \frac{p_{n+1}(x;t)}{t^{(n+1)/2}} + a_{n-1}(t) \frac{c_n(t)}{t} \frac{p_{n-1}(x;t)}{t^{(n-1)/2}}.$$

Now it remains to change variable $x \rightarrow y\sqrt{t}$, use assertion ii) of Lemma 2 and multiply $(\sigma\lambda_{n-1}t + 1)(1 + \tau\lambda_{n-1}/t)$. \square

Now let us recall Theorem 2.5.5 of [15] assuring that the measure that makes polynomials $\{M_n\}$ orthogonal is unique if for every $t > 0$ sequences $\{(\gamma_n t + \delta_n)\}$ and $\{\chi_n(1 + \sigma\lambda_{n-1}t + \tau\lambda_{n-1}/t + \sigma\tau\lambda_{n-1}^2)\}$ are bounded. Notice that following properties of sequence $\{\lambda_n\}$ we deduce that the sequence $\{(1 + \sigma\lambda_{n-1}t + \tau\lambda_{n-1}/t + \sigma\tau\lambda_{n-1}^2)\}$ is bounded and nonnegative for every $t > 0$. Hence the mentioned above theorem requires in fact only that sequences $\{\gamma_n, \delta_n, \chi_n\}$ are bounded. Besides if this measure is to be non-negative then from Favard's Theorem it follows that the sequence $\{\chi_n\}$ has to be additionally nonnegative. Taking into account the fact that from Lemma 2 it follows that if $q > 1 - 2\sqrt{\sigma\tau}$ then the sequence λ_n changes sign infinitely often we deduce that then the sequence $\{\chi_n(1 + \sigma\lambda_{n-1}t + \tau\lambda_n/t + \sigma\tau\lambda_{n-1}\lambda_n)\}$ cannot be non-negative for all $t > 0$. Hence we will consider only the case $q \leq 1 - 2\sqrt{\sigma\tau}$. Proposition below lists several easy cases when almost full solution is possible. The other more complicated cases require separate analysis and treatment.

Proposition 4. *i) If $\tau = \theta = 0$, then $\lambda_n = [n]_q$, $\gamma_n = [n-1]_q\eta$, $\delta_n = 0$, $\chi_n = [n]_q$.*

ii) If $\sigma = \eta = 0$, then $\lambda_n = [n]_q$, $\gamma_n = 0$, $\delta_n = (1 + (-1)^n)\theta/2$, $\chi_n = [n-1]_q$ for $n > 1$ and $\chi_1 = 1$.

iii) If $\sigma = 0$ and $\tau = 0$, then: $\lambda_n = [n]_q$, $\gamma_n = [n-1]_q\eta$, $\delta_n = (1 + (-1)^n)\theta/2$ and $\chi_n = [n]_q + \theta\eta\kappa_n$, where κ_n satisfies recursion: $\kappa_{n+1} = q\kappa_n + [2 \lfloor n/2 \rfloor]_q$ with $\kappa_1 = 1$, for $n > 1$.

iv) If $q = -\sigma\tau$, then $\lambda_n = 1$, $\gamma_n = \begin{cases} \frac{\eta+2\theta\sigma+\eta\sigma\tau}{(1-\sigma\tau)^2} & \text{if } n = 2, 4, \dots \\ \frac{\theta+\eta\tau}{(1-\sigma\tau)^2} & \text{if } n = 1, 3, \dots \end{cases}$, $\delta_n = \begin{cases} \frac{\theta+2\eta\tau+\theta\sigma\tau}{(1-\sigma\tau)^2} & \text{if } n = 2, 4, \dots \\ \frac{(\eta+\theta\sigma)\tau}{(1-\sigma\tau)^2} & \text{if } n = 1, 3, \dots \end{cases}$ and $\beta_1\varepsilon_2 = \frac{1}{(1-\sigma\tau)^2}$ and $\chi_n = \frac{1}{(1-\sigma\tau)^2} + \frac{(\eta+\theta\sigma)(\theta+\eta\tau)}{(1-\sigma\tau)^4}$.

v) If $q = 1 - 2\sqrt{\sigma\tau}$, then $\lambda_n = \frac{n}{1+(n-1)\sqrt{\sigma\tau}}$. Assuming that $\theta = \eta = 0$ sequence $\{\chi_n\}$ is given by the formula:

$$(3.10) \quad \chi_n = \frac{(n-1) + (n-3)n\sqrt{\sigma\tau} + \sigma\tau(1 + (n-2)(n-1)(2n-3)/6)}{(1 + 2n\sqrt{\sigma\tau})}.$$

Proof. Lengthy proof is shifted to Section 6. \square

From the above considerations follows the following Lemma that contains observations concerning polynomials M_n and the distribution that makes these polynomials orthogonal.

Lemma 1. i) If $\theta = \eta = 0$ and $q \leq 1 - 2\sqrt{\sigma\tau}$, then for every $t > 0$ the distribution of X_t is symmetric and positive. Moreover if $q < 1 - 2\sqrt{\sigma\tau}$ this distribution is compactly supported

ii) If $\tau = \theta = 0$, then the polynomials M_n defined by (3.8) satisfy the following 3-term recurrence:

$$yM_n(y) = M_{n+1}(y) + t\eta[n-1]_q M_n(y) + [n-1]_q(1 + \sigma t[n-1]_q)M_{n-1}(y),$$

with $M_{-1}(y) = 0$, $M_0(y) = 1$. That is M_n are in fact modified Al-Salam-Chihara polynomials (see e.g. [15]).

iii) $\sigma = 0$ and $\eta = 0$, then the polynomials M_n defined by (3.8) satisfy the following 3-term recurrence:

$$yM_n(y) = M_{n+1}(y) + (1 + (-1)^n)\theta M_n(y)/2 + [n-1]_q(1 + \tau[n-1]_q/t)M_{n-1}(y),$$

with $M_{-1}(y) = 0$, $M_0(y) = 1$. In particular if $q = 1$ we recognize that polynomials $M_n(y)$ are in fact of Meixner type (see e.g. [15]).

iv) If $\sigma = \tau = 0$, and $|\theta\eta| \leq 1 + q$ then the polynomials M_n defined by (3.8) satisfy the following 3-term recurrence:

$$\begin{aligned} yM_n(y) &= M_{n+1}(y) + [t\eta[n-1]_q + (n-1)\theta]M_n(y) \\ &\quad + ([n]_q + \theta\eta\kappa_n)M_{n-1}(y), \end{aligned}$$

with $M_{-1}(y) = 0$, $M_0(y) = 1$ and define some positive compactly supported measure. In particular if $q = 1$ (that is for $\theta\eta > 0$) we recognize that polynomials $M_n(y)$ are in fact of Meixner type (see e.g. [15]).

v) If $q = -\sigma\tau$ and $(\theta, \eta) \in \{(\theta, \eta) : (1 - \sigma\tau)^2 + \sigma\theta^2 + \tau\eta^2 + \theta\eta(1 + \sigma\tau) \geq 0\}$. Measure that makes polynomials M_n orthogonal is positive and compactly supported.

vi) If $\sigma = \tau = \theta = \eta = 0$ then the polynomials M_n defined by (3.8) satisfy the following 3-term recurrence:

$$yM_n(y) = M_{n+1}(y) + [n-1]_q M_{n-1}(y),$$

that is polynomials are in fact the q -Hermite polynomials (see [15])

Proof. Proof is shifted to Section 6. \square

Remark 1. If $\sigma = \tau = \theta = \eta = 0$ then QH with these parameters is q -Wiener process as described in [17].

4. REMARKS AND OPEN PROBLEMS

First of all notice that the set of allowed values of parameters $\sigma, \tau, \theta, \eta, q$ is such that $\sigma, \tau \geq 0, \theta, \eta \in \mathbb{R}$ and $q \leq 1 + 2\sqrt{\sigma\tau}$. As it follows from the Lemma 2, below if $q \in (1 - 2\sqrt{\sigma\tau}, 1 + 2\sqrt{\sigma\tau}]$ then the sequence $\{\lambda_n\}$ changes sign infinitely often. Hence it is rather unlikely that a set of OMP defining positive 1-dimensional measure exists.

Is it really so? Do there exist QH that are not Markovian and $q > 1 - 2\sqrt{\sigma\tau}$?

The set of OMP of a given QH supplies knowledge about 1-demission distributions. But in fact knowing polynomials of OMP we can also state something about transitional probability distribution. Namely from the relationship (2.7) we can also deduce that orthogonal polynomials $\{W_n\}$ of the transitional probability must be of the form :

$$W_n(X_t, t; X_s, s) = \sum_{j=0}^n V_{n,j}(X_s, s) (a_j(X_t, t) - a_j(X_s, s)),$$

for $s \leq t$ and some polynomials $V_{n,j}(X_s, s)$ of order at most $n - j$. This is so since we have $E_x W_n(X_t, t; x, s) = 0$ for all $n \geq 1$. Thus it remains to prove that these polynomials satisfy some 3-term recurrence to be able to identify them as polynomials orthogonal with respect to the transitional measure.

The examples known so far suggest to seek polynomials $V_{n,j}(X_s, s)$ among such polynomials that :

$$\sum_{j=0}^n V_{n-j}(x; s) a_j(x, s) = 0,$$

for a set of polynomials that are of order $n - j$ and indexed only one integer index.

Can it be true in the general case?

5. AUXILIARY NON-PROBABILISTIC RESULTS

In this section we will analyze properties of some numerical sequences that will appear in examining properties of marginal distributions of MQH. Firstly we will examine properties of the sequence $\{\lambda_n\}$. We have the following observation:

Proposition 5. *Let us denote $f(x|q, z) = \frac{1+qx}{1-zx}$.*

i) If $q + z \geq 0$ then $f(x|q, z) \geq x \geq 0$. In particular if $q + z = 0$ then $f(x|q, z) = 1$.

ii) If $z \in [0, 1)$ and $q \in [-1, 1 - 2\sqrt{z}]$ then for $x \in [0, \frac{1}{\sqrt{z}}]$ implies that $f(x|q, z) \in [0, \frac{1}{\sqrt{z}}]$.

iii) Let $f^{(n)}$ denote n -fold composition of function f . If $q \leq 1 - 2\sqrt{z}$ or $q \geq 1 + 2\sqrt{z}$ then for every n there exists a number y_n such that $y_n = f^{(n)}(y_n|q, z)$. Otherwise such number does not exist. Moreover if they do exist all numbers y_n , are identical and equal to

$$y(q, z) = \begin{cases} \frac{1}{1-q} & \text{if } z = 0 \\ \frac{1-q-\sqrt{(1-q)^2-4z}}{2z} & \text{if } z > 0 \end{cases}.$$

Proof. Uninteresting proof is shifted to Section 6. □

From Proposition 5 there follow the following properties of sequences $\{\alpha_n/\beta_n\}$ and $\{\varphi_n/\varepsilon_n\}$.

Lemma 2. Let $\{\lambda_n\}$ denote number sequence generated by recursion (3.1) with initial condition $\lambda_0 = 0$. then:

- i) If $\tau\sigma = 0$, then $\lambda_n = [n]_q$, $n \geq 1$. If $q + \sigma\tau = 0$ then $\lambda_n = 1$ for $n \geq 1$.
- ii) $\frac{\alpha_n}{\beta_n} = \sigma\lambda_n$ and $\frac{\varphi_n}{\varepsilon_n} = \tau\lambda_{n-1}$, consequently coefficients $a_n(t)$ and $c_n(t)$ are equal respectively to $\beta_n(\sigma\lambda_n t + 1)$ and $\varepsilon_n(t + \tau\lambda_n)$
- iii) If $q \leq 1 - 2\sqrt{\sigma\tau}$ then sequence λ_n is nonnegative and has one condensation point equal to

$$y(q, z) = \begin{cases} \frac{1}{1-q} & \text{if } \sigma\tau = 0 \\ \frac{1-q-\sqrt{(1-q)^2-4\sigma\tau}}{2\sigma\tau} & \text{if } \sigma\tau > 0 \end{cases},$$

if additionally $q + \sigma\tau \geq 0$ then the sequence $\{\lambda_n\}$ is non-decreasing.

- iv) If $q > 1 - 2\sqrt{\sigma\tau}$ then sequence $\{\lambda_n\}$ changes sign infinitely many times.

Proof. i) If either σ or τ is equal zero then $\lambda_{n+1} = 1 + q\lambda_n$, $n \geq 1$, with $\lambda_0 = 0$. If $q + \sigma\tau = 0$ then $\lambda_{n+1} = 1$ for $n > 1$ by assertion i) of Proposition 5. ii) The fact that $\frac{\alpha_n}{\beta_n} = \sigma\lambda_n$ was shown above. To prove similar statement concerning ratio $\frac{\varphi_n}{\varepsilon_n}$ we consider equation (2.10). Let us divide both sides by $\varepsilon_{n-1}\varepsilon_n$ and denote $\varphi_n/\varepsilon_n = f_n$. Thus

$$f_{n+1} = \frac{\tau + qf_n}{1 - \sigma f_n},$$

with $f_1 = 0$. If $\tau = 0$ then $f_n = 0$ for $n \geq 1$. Then we argue in the similar way as in the case of the sequence d_n . iii) We use assertions ii) and iii) of Proposition 5 deducing that sequence $\{\lambda_n\}$ is nonnegative and bounded (by $\frac{1}{\sqrt{\sigma\tau}}$). iv) If $q > 1 - 2\sqrt{\sigma\tau}$, then as it follows from assertion iii) of Proposition 5 there is no condensation point of the sequence $\{\lambda_n\}$. Since in this case we have $q + z \geq 0$ then the sequence $\{\lambda_n\}$ is increasing and consequently will reach value more than $\frac{1}{z}$. But then the next iterate will be negative and again the sequence will be increasing and so on. \square

Let us also analyze function

$$g(x|q, z) = \frac{q + z(2x - x^2)}{1 - z(2x + qx^2)}.$$

We have the following proposition.

Proposition 6. If $q < 1 - 2\sqrt{z}$ then for $x \in [0, 1/\sqrt{z}]$: $g(x|q, z) < 1$.

Proof. First of all let us introduce new parameter $t = (1 - q)^2/4 - z$. Hence $z = (1 - q)^2/4 - t$ and $\frac{1}{\sqrt{z}} = \frac{2}{\sqrt{(1 - q)^2 - 4t}}$. Function g with this new parameter has the following form:

$$\begin{aligned} h(x|q, t) &= g(x|q, (1 - q)^2/4 - t) \\ &= \frac{4q - x(x - 2)((1 - q)^2 - 4t)}{4 - x(2 + qx)((1 - q)^2 - 4t)}. \end{aligned}$$

Secondly let us find roots of the derivative of $h(x|q, t)$ with respect to x . By direct calculation we get that they are equal to $\frac{2}{(1 - q) + 2\sqrt{t}}$ and $\frac{2}{(1 - q) - 2\sqrt{t}}$. Since notice that $\frac{2}{(1 - q) + 2\sqrt{t}} \leq \frac{2}{\sqrt{(1 - q)^2 - 4t}}$ since this inequality is equivalent to the following one:

$$\sqrt{\frac{(1 - q) - 2t}{(1 - q) + 2t}} \leq 1.$$

which is obviously true for $t \in [0, (1-q)^2/4]$. Similarly we show that $\frac{2}{(1-q)-2\sqrt{t}} > \frac{2}{\sqrt{(1-q)^2-4t}}$. Hence we deduce that on $[0, 2/\sqrt{(1-q)^2-4t}]$ function g has only one maximum equal to $h(\frac{2}{(1-q)+2\sqrt{t}}|q, z) = \frac{1-q-2\sqrt{t}}{1-q+2\sqrt{t}} < 1$ for $t > 0$ which is equivalent to $z < (1-q)^2/4$. \square

6. PROOFS

Proof of Proposition 1. i) First of all notice that from (2.7) it follows that $\forall n > -1$ $Ep_n(X_t, t) = Ep_n(0, 0) = \xi_n$ a constant that does not depend on t . Secondly notice that $a_n(0) = \beta_n$, $b_n(0) = \delta_n$, $c_n(0) = \varphi_n$. Further notice that following (2.8) these constants satisfy the following second order recursion:

$$\xi_{n+1} = -\frac{\delta_n}{\beta_n}\xi_n - \frac{\varphi_n}{\beta_n}\xi_{n-1},$$

with $\xi_{-1} = 0$, $\xi_0 = 1$. Besides we also have $0 = \beta_0 p_1(0; 0) + \delta_0 p_0(0; 0) + \varphi_0 p_{-1}(0; 0)$. Hence we deduce that $p_1(0, 0) = 0$, that is $\xi_1 = 0$. Now notice that if we chose $\varphi_1 = 0$ then we would have $\xi_2 = 0$ that is two successive constants ξ_n being equal to zero consequently all must be equal to zero. Thus the choice $\varphi_1 = 0$ enables to select sequence $\{p_n\}$ to be both OM and have the property that $Ep_n(X_t, t) = 0$. On the other hand since $EX_t^2 = t$ we take $n = 1$ in (2.8) and use the fact that $Ep_n(X_t, t) = 0$ and deducing that $\varepsilon_1 = 1$. Sequence $\{p_n\}$ is thus a sequence of orthogonal polynomials that for some measure μ satisfy $\int p_n d\mu = 0$ for all $n > 0$. Since we have also 3-term recurrence satisfied by polynomials p_n we deduce that also $\int xp_n d\mu = 0$ for all $n > 1$. Similarly we deduce that $\int x^k p_n d\mu = 0$ for all $n > k$. Hence polynomials must constitute family of orthogonal polynomials of measure μ .

ii) On one hand we have: $\mathbb{E}(X_t p_n(X_t; t) | \mathcal{B}_{\leq s}) = a_n(t)p_{n+1}(X_s; s) + b_n(t)p_n(X_s; s) + c_n(t)p_{n-1}(X_s; s)$. On the other:

$$\begin{aligned} \mathbb{E}(X_t p_n(X_t; t) | \mathcal{B}_{\leq s}) &= \mathbb{E}(X_t p_n(X_u; u) | \mathcal{B}_{\leq s}) = \mathbb{E}(\mathbb{E}(X_t | \mathcal{B}_{\leq s, \geq u}) p_n(X_u; u) | \mathcal{B}_{\leq s}) = \\ &= \frac{(u-t)}{u-s} X_s p_n(X_s; s) + \frac{t-s}{u-s} \mathbb{E}(X_u p_n(X_u; u) | \mathcal{B}_{\leq s}) \\ &= \frac{(u-t)}{u-s} (a_n(s)p_{n+1}(X_s; s) + b_n(s)p_n(X_s; s) + c_n(s)p_{n-1}(X_s; s)) \\ &\quad + \frac{t-s}{u-s} (a_n(u)p_{n+1}(X_s; s) + b_n(u)p_n(X_s; s) + c_n(u)p_{n-1}(X_s; s)). \end{aligned}$$

Comparing appropriate coefficients we get: $a_n(t) = \frac{(u-t)}{u-s} a_n(s) + \frac{t-s}{u-s} a_n(u)$ and similarly for $b_n(t)$ and $c_n(t)$. Under assumption of continuity of $a_n(t)$ the solution must be a linear function of t : $a_n(t) = \alpha_n t + \beta_n$ similarly for other coefficients b_n and c_n . \square

Proof of Theorem 1. First of all notice that starting from : $x p_n(x; t) = (\alpha_n t + \beta_n) p_{n+1}(x; t) + (\gamma_n t + \delta_n) p_n(x; t) + (\varepsilon_n t + \varphi_n) p_{n-1}(x; t)$

we get:

$$\begin{aligned}
x^2 p_n(x; t) &= (\alpha_n t + \beta_n)((\alpha_{n+1} t + \beta_{n+1})p_{n+2}(x; t) \\
&+ (\gamma_{n+1} t + \delta_{n+1})p_{n+1}(x; t) + (\varepsilon_{n+1} t + \varphi_{n+1})p_n(x; t)) \\
&+ (\gamma_n t + \delta_n)((\alpha_n t + \beta_n)p_{n+1}(x; t) + (\gamma_n t + \delta_n)p_n(x; t) \\
&+ (\varepsilon_n t + \varphi_n)p_{n-1}(x; t)) + (\varepsilon_n t + \varphi_n)((\alpha_{n-1} t + \beta_{n-1})p_n(x; t) \\
&+ (\gamma_{n-1} t + \delta_{n-1})p_{n-1}(x; t) + (\varepsilon_{n-1} t + \varphi_{n-1})p_{n-2}(x; t)) = \\
&(\alpha_n t + \beta_n)(\alpha_{n+1} t + \beta_{n+1})p_{n+2}(x; t) + ((\alpha_n t + \beta_n)(\gamma_{n+1} t + \delta_{n+1}) \\
&+ (\gamma_n t + \delta_n)(\alpha_n t + \beta_n))p_{n+1}(x; t) \\
&+ ((\alpha_n t + \beta_n)(\varepsilon_{n+1} t + \varphi_{n+1}) + (\gamma_n t + \delta_n)(\gamma_n t + \delta_n) \\
&+ (\varepsilon_n t + \varphi_n)(\alpha_{n-1} t + \beta_{n-1}))p_n(x; t) \\
&+ ((\gamma_n t + \delta_n)(\varepsilon_n t + \varphi_n) + (\varepsilon_n t + \varphi_n)(\gamma_{n-1} t + \delta_{n-1}))p_{n-1}(x; t) \\
&+ (\varepsilon_n t + \varphi_n)(\varepsilon_{n-1} t + \varphi_{n-1})p_{n-2}(x; t).
\end{aligned}$$

On one hand we get:

$$\begin{aligned}
\mathbb{E}(X_t^2 p_n(X_t; t) | \mathcal{B}_{\leq s}) &= (\alpha_n t + \beta_n)(\alpha_{n+1} t + \beta_{n+1})p_{n+2}(X_s; s) \\
&+ ((\alpha_n t + \beta_n)(\gamma_{n+1} t + \delta_{n+1}) + (\gamma_n t + \delta_n)(\alpha_n t + \beta_n))p_{n+1}(X_s; s) \\
&+ ((\alpha_n t + \beta_n)(\varepsilon_{n+1} t + \varphi_{n+1}) + (\gamma_n t + \delta_n)(\gamma_n t + \delta_n) + (\varepsilon_n t + \varphi_n)(\alpha_{n-1} t + \beta_{n-1}))p_n(X_s; s) \\
&+ ((\gamma_n t + \delta_n)(\varepsilon_n t + \varphi_n) + (\varepsilon_n t + \varphi_n)(\gamma_{n-1} t + \delta_{n-1}))p_{n-1}(X_s; s) \\
&+ (\varepsilon_n t + \varphi_n)(\varepsilon_{n-1} t + \varphi_{n-1})p_{n-2}(X_s; s).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X_t^2 p_n(X_t; t) | \mathcal{B}_{\leq s}) &= \mathbb{E}(X_t^2 p_n(X_u; u) | \mathcal{B}_{\leq s}) = \mathbb{E}(\mathbb{E}(X_t^2 | \mathcal{B}_{\leq s, \geq u}) p_n(X_u; u) | \mathcal{B}_{\leq s}) \\
&= \mathbb{E}(((A(s, t, u)X_s^2 + B(s, t, u)X_s X_u + C(s, t, u)X_u^2 + D(s, t, u)X_s \\
&\quad + E(s, t, u)X_u + F(s, t, u))p_n(X_u; u) | \mathcal{B}_{\leq s}) \\
&= A(s, t, u)X_s^2 p_n(X_s; s) + B(s, t, u)X_s((\alpha_n u + \beta_n)p_{n+1}(X_s; s) \\
&\quad + (\gamma_n u + \delta_n)p_n(X_s; s) + (\varepsilon_n u + \varphi_n)p_{n-1}(X_s; s)) \\
&\quad + C(s, t, u)((\alpha_n u + \beta_n)(\alpha_{n+1} u + \beta_{n+1})p_{n+2}(X_s; s) \\
&\quad + ((\alpha_n u + \beta_n)(\gamma_{n+1} u + \delta_{n+1}) + (\gamma_n u + \delta_n)(\alpha_n u + \beta_n))p_{n+1}(X_s; s) \\
&\quad + ((\alpha_n u + \beta_n)(\varepsilon_{n+1} u + \varphi_{n+1}) + (\gamma_n u + \delta_n)(\gamma_n u + \delta_n) \\
&\quad + (\varepsilon_n u + \varphi_n)(\alpha_{n-1} u + \beta_{n-1}))p_n(X_s; s) \\
&\quad + ((\gamma_n u + \delta_n)(\varepsilon_n u + \varphi_n) + (\varepsilon_n u + \varphi_n)(\gamma_{n-1} u + \delta_{n-1}))p_{n-1}(X_s; s) \\
&\quad + (\varepsilon_n u + \varphi_n)(\varepsilon_{n-1} u + \varphi_{n-1})p_{n-2}(X_s; s)) \\
&\quad + D(s, t, u)X_s p_n(X_s; s) + E(s, t, u)((\alpha_n u + \beta_n)p_{n+1}(X_s; s) \\
&\quad + (\gamma_n u + \delta_n)p_n(X_s; s) + (\varepsilon_n u + \varphi_n)p_{n-1}(X_s; s) + F(s, t, u)p_n(X_s; s).
\end{aligned}$$

Comparing coefficients by respectively $p_{n+i}(X_s; s)$, $i = 2, -2, 1, -1, 0$ we get:

$$\begin{aligned}
0 &= (\alpha_n t + \beta_n)(\alpha_{n+1} t + \beta_{n+1}) - A(s, t, u)(\alpha_n s + \beta_n)((\alpha_{n+1} s + \beta_{n+1}) \\
&\quad - B(s, t, u)(\alpha_n u + \beta_n)(\alpha_{n+1} s + \beta_n) - C(s, t, u)(\alpha_n u + \beta_n)(\alpha_{n+1} u + \beta_{n+1})
\end{aligned}$$

from which it follows that (by Mathematica):

$$\tau \alpha_n \alpha_{n+1} - \alpha_{n+1} \beta_n + q \alpha_n \beta_{n+1} + \sigma \beta_n \beta_{n+1} = 0.$$

$$\begin{aligned}
0 = & (\varepsilon_n t + \varphi_n)(\varepsilon_{n-1} t + \varphi_{n-1}) - B(s, t, u)(\varepsilon_n u + \varphi_n)(\varepsilon_{n-1} s + \varphi_{n-1}) \\
& - C(s, t, u)(\varepsilon_n u + \varphi_n)(\varepsilon_{n-1} u + \varphi_{n-1}) - A(s, t, u)(\varepsilon_n s + \varphi_n)(\varepsilon_{n-1} s + \varphi_{n-1})
\end{aligned}$$

from which it follows that (by Mathematica):

$$\tau \varepsilon_n \varepsilon_{n-1} + q \varepsilon_n \varphi_{n-1} - \varepsilon_{n-1} \varphi_n + \sigma \varphi_{n-1} \varphi_n = 0.$$

$$\begin{aligned}
0 = & (\alpha_n t + \beta_n)(\gamma_n t + \delta_n + \gamma_{n+1} t + \delta_{n+1}) \\
& - A(s, t, u)(\alpha_n s + \beta_n)(\gamma_n s + \delta_n + \gamma_{n+1} s + \delta_{n+1}) \\
& - B(s, t, u)((\alpha_n u + \beta_n)(\gamma_{n+1} s + \delta_{n+1}) + (\alpha_n s + \beta_n)(\gamma_n s + \delta_n)) \\
& - C(s, t, u)(\alpha_n u + \beta_n)(\gamma_n u + \delta_n + \gamma_{n+1} u + \delta_{n+1}) \\
& - D(s, t, u)(\alpha_n s + \beta_n) - E(s, t, u)(\alpha_n u + \beta_n)
\end{aligned}$$

from which it follows by Mathematica:

$$\theta \alpha_n + \eta \beta_n + \tau \alpha_n (\gamma_n + \gamma_{n+1}) + q \beta_n \gamma_n - \beta_n \gamma_{n+1} - \alpha_n \delta_n + \sigma \beta_n (\delta_n + \delta_{n+1}) + q \alpha_n \delta_{n+1} = 0.$$

Further :

$$\begin{aligned}
0 = & (\gamma_n t + \delta_n)(\varepsilon_n t + \varphi_n) + (\varepsilon_n t + \varphi_n)(\gamma_{n-1} t + \delta_{n-1}) \\
& - A(s, t, u)(\varepsilon_n s + \varphi_n)((\gamma_n s + \delta_n) + (\gamma_{n-1} s + \delta_{n-1})) \\
& - B(s, t, u)((\gamma_n u + \delta_n)(\varepsilon_n s + \varphi_n) + (\varepsilon_n u + \varphi_n)(\gamma_{n-1} s + \delta_{n-1})) \\
& - C(s, t, u)(\varepsilon_n u + \varphi_n)((\gamma_n u + \delta_n) + (\gamma_{n-1} u + \delta_{n-1})) \\
& - D(s, t, u)(\varepsilon_n s + \varphi_n) - E(s, t, u)(\varepsilon_n u + \varphi_n)
\end{aligned}$$

from which it follows by Mathematica:

$$\theta \varepsilon_n + \eta \varphi_n + \tau \varepsilon_n (\gamma_n + \gamma_{n-1}) - \varepsilon_n (\delta_n + \delta_{n-1}) + q \varphi_n (\gamma_n + \gamma_{n-1}) + \sigma \varphi_n (\delta_n + \delta_{n-1}) = 0.$$

Finally:

$$\begin{aligned}
0 = & (\alpha_n t + \beta_n)(\varepsilon_{n+1} t + \varphi_{n+1}) + (\gamma_n t + \delta_n)(\gamma_n t + \delta_n) + (\varepsilon_n t + \varphi_n)(\alpha_{n-1} t + \beta_{n-1}) \\
& - A(s, t, u)((\alpha_n s + \beta_n)(\varepsilon_{n+1} s + \varphi_{n+1}) \\
& + (\gamma_n s + \delta_n)(\gamma_n s + \delta_n) + (\varepsilon_n s + \varphi_n)(\alpha_{n-1} s + \beta_{n-1})) \\
& - B(s, t, u)((\alpha_n u + \beta_n)(\varepsilon_{n+1} s + \varphi_{n+1}) \\
& + (\gamma_n u + \delta_n)(\gamma_n s + \delta_n) + (\varepsilon_n u + \varphi_n)(\alpha_{n-1} s + \beta_{n-1})) \\
& - C(s, t, u)((\alpha_n u + \beta_n)(\varepsilon_{n+1} u + \varphi_{n+1}) \\
& + (\gamma_n u + \delta_n)(\gamma_n u + \delta_n) + (\varepsilon_n u + \varphi_n)(\alpha_{n-1} u + \beta_{n-1})) \\
& - D(s, t, u)(\gamma_n s + \delta_n) - E(s, t, u)(\gamma_n u + \delta_n) - F(s, t, u)
\end{aligned}$$

from which it follows by Mathematica

$$\begin{aligned}
0 = & 1 + \theta \gamma_n + \eta \delta_n + \tau \gamma_n^2 + \sigma \delta_n^2 - (1 - q) \gamma_n \delta_n + \tau (\alpha_{n-1} \varepsilon_n + \alpha_n \varepsilon_{n+1}) \\
& + q \beta_{n-1} \varepsilon_n - \beta_n \varepsilon_{n+1} - \alpha_{n-1} \varphi_n + \sigma (\beta_{n-1} \varphi_n + \beta_n \varphi_{n+1}) + q \alpha_n \varphi_{n+1}.
\end{aligned}$$

□

Proof of Proposition 5. i) If $q + z \geq 0$ then the derivative of the function $f(x)$ is nonnegative, hence the first assertion is true. ii) If $q + z < 0$ then we have for $\frac{1}{\sqrt{z}}$
 $\geq x \geq 0$ $f(x) = 1 + \frac{(q+z)x}{(1-xz)} \leq 1 \leq \frac{1}{\sqrt{z}}$. If $q + z \geq 0$ which is equivalent to $(q + z) \leq (1 - \sqrt{z})^2$ then $f(x)$ is non-decreasing and we have $f(x) \leq f\left(\frac{1}{\sqrt{z}}\right) = 1 + \frac{(q+z)}{\sqrt{z}(1-\sqrt{z})}$

$\leq 1 + \frac{1-\sqrt{z}}{\sqrt{z}} = \frac{1}{\sqrt{z}}$. iii) If $z = 0$ then $f(x) = 1 + qx$, hence $f^{(n)}(x) = [n-1]_q + q^n x$, consequently $y_n = \frac{1}{1-q}$. Assume that $z \neq 0$. Let us notice that

$$f^{(n)}(x) = \frac{A_n + B_n x}{C_n - D_n x},$$

for some depending on q and z functions A_n, B_n, C_n, D_n . Notice that the solution of the equation $f^{(n)}(y_n) = y_n$ satisfies the quadratic equation:

$$D_n y^2 + (B_n - C_n)y + A_n = 0.$$

Since $f(f^{(n)}(x)) = f^{(n)}(f(x))$ for every x we deduce that:

$$\begin{aligned} A_n + B_n &= qA_n + C_n, \\ -zA_n + qB_n &= qB_n - D_n \\ C_n - D_n &= C_n - zA_n, \\ C_n z + qD_n &= D_n + zB_n, \end{aligned}$$

and consequently that $B_n - C_n = (1-q)A_n$ and $D_n = zA_n$. Since $A_n \neq 0$ (otherwise we would have $f^{(n)}(x) \equiv x$) we deduce that for all n number y_n satisfies equation

$$zy^2 + (1-q)y + 1 = 0.$$

Moreover real solution of this equation exists if $(1-q)^2 \geq 4z$ or equivalently if $q \leq 1 - 2\sqrt{z}$ or $q \geq 1 + 2\sqrt{z}$. Now let us consider the case $q \in (1 - 2\sqrt{z}, 1 + 2\sqrt{z})$. Then as the above analysis shows there is no solution of the equation $f^{(n)}(x) = x$ for any $n > 0$. \square

Proof of Proposition 4. First of all notice that if $\sigma\tau = 0$ then $\lambda_n = [n]_q$ since then equation (3.1) reduces to $\lambda_{n+1} = q\lambda_n + 1$, with $\lambda_0 = 0$. i) Under our assumptions we get $A_n = \begin{bmatrix} 1 & -\sigma[n+1]_q \\ 0 & 1 \end{bmatrix}$, $B_n = \begin{bmatrix} q & -\sigma q[n-1]_q \\ 0 & -1 \end{bmatrix}$ and $C_n =$

$\begin{bmatrix} \sigma[n]_n & 1 \\ 1 & 0 \end{bmatrix}$ since $1 + q[n]_q = [n+1]_q$ and $1 - [n]_q = -q[n-1]_q$. So $A_n^{-1}B_n = \begin{bmatrix} q & -\sigma(1+q)[n]_n \\ 0 & -1 \end{bmatrix}$ and $A_n^{-1}C_n \begin{bmatrix} 0 \\ \eta \end{bmatrix} = \begin{bmatrix} \eta \\ 0 \end{bmatrix}$. Hence vector $\begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix}$ satisfies the following recursion: $\begin{bmatrix} \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = \begin{bmatrix} q & -\sigma(1+q)[n]_q \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix} + \begin{bmatrix} \eta \\ 0 \end{bmatrix}$. Hence $\delta_n = 0$, while $\gamma_n = [n+1]_q$. Further we have

$1 + \theta\gamma_n + \tau\gamma_n^2 + \eta\delta_n + \sigma\delta_n^2 - (1-q)\gamma_n\delta_n|_{\tau=0, \theta=0, \gamma_n=\eta[n-1]_q, \delta_n=0} = 1$, so recursion (3.7) reduces to

$$q\chi_n + 1 = \chi_{n+1},$$

with $\chi_1 = 1$. Thus indeed $\chi_n = [n]_q$.

ii) Under our assumptions we get $A_n = \begin{bmatrix} 1 & 0 \\ -\tau[n+1]_q & 1 \end{bmatrix}$, $B_n = \begin{bmatrix} q & 0 \\ \tau[n+1]_q & -1 \end{bmatrix}$ and $C_n = \begin{bmatrix} 0 & 1 \\ 1 & \tau[n]_q \end{bmatrix}$. Hence $A_n^{-1}B_n = \begin{bmatrix} q & 0 \\ (1+q)[n+1]_q\tau & -1 \end{bmatrix}$, $A_n^{-1}C_n \begin{bmatrix} \theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta \end{bmatrix}$. So $\gamma_n = 0$ and $\delta_n = (1 + (-1)^n)\theta/2$. Further we have

$1 + \theta\gamma_n + \tau\gamma_n^2 + \eta\delta_n + \sigma\delta_n^2 - (1-q)\gamma_n\delta_n|_{\sigma=0, \eta=0, \gamma_n=0, \delta_n=0} = 1$ and so recursion (3.7) reduces to

$$q\chi_n + 1 = \chi_{n+1},$$

with $\chi_1 = 1$. Thus indeed $\chi_n = [n]_q$.

iii) Under our assumptions we have $A_n = I$, $B_n = \begin{bmatrix} q & 0 \\ 0 & -1 \end{bmatrix}$, $C_n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Vector $\begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix}$ satisfies then 2 separate equations : $\gamma_{n+1} = q\gamma_n + \eta$ which results in $\gamma_n = \eta[n-1]_q$ and $\delta_{n+1} = -\delta_n + \theta$. Which results in $(1 + (-1)^n)\theta/2$. Now Inserting these quantities to equation (3.7) yields the following recursion:

$$\begin{aligned} \chi_{n+1} &= q\chi_n + 1 + \theta\eta([n-1]_q + (1 + (-1)^n)/2 - (1-q)(1 + (-1)^n)[n-1]_q/2) \\ &= q\chi_n + 1 + \theta\eta[2[n/2]]_q. \end{aligned}$$

iv) Under this assumption equation (3.1) reduces to

$$\lambda_{n+1} = \frac{1 - \sigma\tau\lambda_n}{1 - \sigma\tau\lambda_n} = 1.$$

Besides we have: $A_n = \begin{bmatrix} 1 - \tau\sigma & -\sigma(1 - \sigma\tau) \\ -\tau(1 - \sigma\tau) & 1 - \sigma\tau \end{bmatrix}$, $B_n = \begin{bmatrix} 0 & 0 \\ \tau(1 - \sigma\tau) & -1 + \sigma\tau \end{bmatrix}$,
 $C_n = C_n = \begin{bmatrix} \sigma & 1 \\ 1 & \tau \end{bmatrix}$, so $\Xi_n = A_n^{-1}B_n = \begin{bmatrix} \frac{\sigma\tau}{1-\sigma\tau} & -\frac{\sigma}{1-\sigma\tau} \\ \frac{\tau}{1-\sigma\tau} & -\frac{1}{1-\sigma\tau} \end{bmatrix} \stackrel{df}{=} \Xi$, Further
 $\prod_{k=1}^n \Xi_k = (-1)^n \Xi$, $w_n = \frac{1}{(1-\sigma\tau)^2} \begin{bmatrix} 2\theta\sigma + \eta(1 + \sigma\tau) \\ 2\eta\tau + \theta(1 + \sigma\tau) \end{bmatrix}$, $\Xi w_n = \frac{\eta + \theta\sigma}{(1-\sigma\tau)^2} \begin{bmatrix} 1 \\ \tau \end{bmatrix}$ So
 $\begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix} = \frac{1}{(1-\sigma\tau)^2} \begin{bmatrix} 2\theta\sigma + \eta(1 + \sigma\tau) \\ 2\eta\tau + \theta(1 + \sigma\tau) \end{bmatrix}$ if $n = 1, 3, 5, \dots$ and $\frac{\eta + \theta\sigma}{(1-\sigma\tau)^2} \begin{bmatrix} 1 \\ \tau \end{bmatrix}$ for $n = 2, 4, \dots$. Besides $(q + \sigma\tau - \sigma\tau(1 - \lambda_n)^2)|_{q+\sigma\tau=0} = 0$ and $(1 - \sigma\tau(2\lambda_n + q\lambda_n^2))|_{q+\sigma\tau=0} = (1 - \sigma\tau)^2$. Besides since γ_n and δ_n do not depend on n we deduce that $\beta_{n-1}\varepsilon_n$ also.

v) First of all notice that under our assumptions we have $q + \sigma\tau = (1 - \sqrt{\sigma\tau})^2$. Next notice that if $n = 1$ then $\lambda_1 = 1 = \frac{n}{1+(n-1)\sqrt{\sigma\tau}}|_{n=1}$. Hence by induction we have $1 + (1 - 2\sqrt{\sigma\tau})n/(1 + (n-1)\sqrt{\sigma\tau}) = \frac{(1-\sqrt{\sigma\tau})(n+1)}{1+(n-1)\sqrt{\sigma\tau}}$ and $1 - \sigma\tau n/(1 + (n-1)\sqrt{\sigma\tau}) = \frac{(1-\sqrt{\sigma\tau})(1+n\sqrt{\sigma\tau})}{(1+(n-1)\sqrt{\sigma\tau})}$. Hence

$$\lambda_{n+1} = (1 + q\lambda_n)/(1 - \sigma\tau\lambda_n)|_{\lambda_n=n/(1+(n-1)\sqrt{\sigma\tau})} = \frac{n+1}{1+n\sqrt{\sigma\tau}}.$$

Now notice that $q + \sigma\tau - \sigma\tau(1 - \lambda_n)^2|_{q=1-2\sqrt{\sigma\tau}} = (1 - \sqrt{\sigma\tau})^2 - \sigma\tau(1 - \frac{n}{1+(n-1)\sqrt{\sigma\tau}})^2 = \frac{(1-\sqrt{\sigma\tau})^2(1+2(n-1)\sqrt{\sigma\tau})}{(1+(n-1)\sqrt{\sigma\tau})^2}$ and $(1 - \sigma\tau(2\lambda_n + q\lambda_n^2))|_{q=1-2\sqrt{\sigma\tau}} = \frac{(1-\sqrt{\sigma\tau})^2(1+2n\sqrt{\sigma\tau})}{(1+(n-1)\sqrt{\sigma\tau})^2}$. So sequence $\{\chi_n\}$ satisfies the following recursion:

$$(6.1) \quad \chi_{n+1} = \frac{(1 + 2(n-1)\sqrt{\sigma\tau})}{(1 + 2n\sqrt{\sigma\tau})} \chi_n + \frac{(1 + (n-1)\sqrt{\sigma\tau})^2}{(1 - \sqrt{\sigma\tau})^2(1 + 2n\sqrt{\sigma\tau})}.$$

Denoting $\Lambda_n = \chi_n(1 + 2(n-1)\sqrt{\sigma\tau})(1 - \sqrt{\sigma\tau})^2$ and multiplying both sides of (6.1) by $(1 + 2n\sqrt{\sigma\tau})(1 - \sqrt{\sigma\tau})^2$ we get:

$$\Lambda_{n+1} = \Lambda_n + (1 + (n-1)\sqrt{\sigma\tau})^2.$$

Hence $\Lambda_n = (1 - \sqrt{\sigma\tau})^2 + \sum_{j=1}^n (1 + (j-1)\sqrt{\sigma\tau})^2$ for $n > 1$. Remembering that $\sum_{j=1}^n j = n(n+1)/2$ and $\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6$ we get (3.10). \square

Proof of Lemma 1. i) For the case $q < 1 - 2\sqrt{\sigma\tau}$ we use Proposition 6. To do this notice that ratio coefficients by respectively χ_n and χ_{n+1} is exactly equal to function $g(\lambda_n|q, \sigma\tau)$ since $\lambda_{n+1} = f(\lambda_n|q, \sigma\tau)$. By Proposition 5 and Lemma 2 we know that under our assumption concerning parameters $q, \sigma, \tau \forall n \geq 1 : \lambda_n \leq \frac{1}{\sqrt{\sigma\tau}}$, so assumptions of Proposition 6 are satisfied.

On the other hand since by assumption sequences $\{\gamma_n\}$ and $\{\delta_n\}$ are bounded then so is the sequence $\{1 + \theta\gamma_n + \tau\gamma_n^2 + \eta\delta_n + \tau\delta_n^2 - (1-q)\gamma_n\delta_n\}$. Now notice also that the sequence $\left\{\frac{1}{(1-\sigma\tau(2\lambda_n+q\lambda_n^2))}\right\}$ is also bounded since

$$\begin{aligned} \frac{1}{(1-\sigma\tau(2\lambda_n+q\lambda_n^2))} &\leq \frac{1}{1-\sigma\tau(2/\sqrt{\sigma\tau}+q/\sigma\tau)} \\ &= \frac{1}{1-q-2\sqrt{\sigma\tau}} < \infty \end{aligned}$$

under our assumptions. Finally we use commonly known result of differential equations that stable system excited by the bounded input is stable. Stability is guaranteed by the assertion of Proposition 6, boundedness of the input was shown above thus output of our system i.e. sequence $\{\beta_{n-1}\varepsilon_n\}$ is stable that is bounded. To get second part of this assertion notice that the fact that $\gamma_n = \delta_n = 0$ follows directly (3.5) and the fact that then this equation reduces to the following

$$\begin{bmatrix} \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = A_n^{-1} B_n \begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix},$$

with initial condition $\begin{bmatrix} \gamma_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. For the the case $q = 1 - 2\sqrt{\sigma\tau}$ we use assertion vi) of Proposition 4.

ii)-vi) follows directly Proposition 4. The only thing that requires comment is classification of orthogonal polynomials. We call Meixner polynomials that have the following 3-term recurrence:

$$xM_n(x) = M_{n+1}(x) + p_nM_n(x) + q_nM_{n-1}(x),$$

where p_n is linear in n while q_n is quadratic, while by modified Al-Salam–Chihara we mean polynomials with the following 3-term recurrence:

$$xA_n(x) = A_{n+1}(x) + \tilde{p}_nA_n(x) + \tilde{q}_nA_{n-1}(x),$$

where \tilde{p}_n is linear function of $(1 - q^n)$ while \tilde{q}_n is quadratic in $(1 - q^n)$.

Note also that we have to assure that the sequence $\{\chi_n\}$ is nonnegative. First of all notice that for $n \geq 2 : [2 \lfloor n/2 \rfloor]_q > (1+q)$. Now notice that we have $\kappa_{n+1} \geq q\kappa_n + (1+q)$ which gives $\kappa_n \geq (1+q)[n-2]_q = [n-1]_q$. Consequently $[n]_q + \theta\eta\kappa_n \geq [n]_q - |\theta\eta|[n-1]_q = [n-1]_q + [n-1]_q(q - |\theta\eta|) \geq 0$ if only $1+q \geq |\theta\eta|$. \square

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